Study of Some Functions with Their Applications Using Weird Numbers (Zwain Weirdness Filter)

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Abstract

A detailed analytical approach, using the structural study of Euler's function ,φ (n), and the divisor sum function, σ (n), is developed in this work to investigate the property of weird numbers. Here, multiples that fail to be smooth are termed weird. Weird numbers are a special class with two basic properties: abundance and non-semi perfectness. A positive integer is called abundant if the sum of its proper factors is greater than the number; otherwise, it is called weird if no subset of proper factors adds up to the number. The identification of weird numbers has traditionally depended on computationally expensive cursive subset-sum computations, not breaking any numbers down into odd chunks; it is impractical to efficiently break up large integers. This can be compared for greater/lesser or equal for integers of different magnitudes. We focus on integers of the form $n = 2^k p q$ and extend to $n = 2^k pq_1 q_2$. Explicit formulas for the function φ (n) and the function σ (n), and the change of these in multiplicative form into the form of asymptotic with some error bounds are found. A significant one is the filtering quantity, which is an initial sign of oddness. Based on the numerical results, the accuracy of the approximations is verified, and the effectiveness of the proposed filter is ensured, which is a framework that can be extended to detect odd numbers in a scalable manner.

Keywords: Euler's function, sum of divisors function, weird numbers, imperfect numbers.

دراسة بعض الدوال وتطبيقاتها باستخدام الأعداد الغريبة (مرشح زوين للغرابة)

هند مهدي زوين
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الملخص

يطور هذا العمل منهج تحليلي مفصل، يعتمد على الدراسة البنيوية لدالة اولير($\boldsymbol{\varphi(n)}$) ودالة مجموع القواسم $\boldsymbol{\sigma(n)}$، بهدف دراسة خاصية الاعداد الغريبة. تعد الاعداد الغريبة فئة خاصة من الأعداد، تتميز بخاصيتين أساسيتين هما: الوفرة وعدم شبه الكمال. ويسمى العدد الصحيح الموجب وفيرا إذا كان مجموع قواسمه الصحيحة الموجبة باستثناء العدد أكبر منه، بينما يسمى العدد الصحيح غريبا إذا لم توجد مجموعة جزئية من قواسمه الصحيحة الموجبة يكون مجموع عناصرها مساويا للعدد نفسه. وقد اعتمد تحديد الاعداد الغريبة تقليديا على عمليات حسابية متكررة ومكلفة من الناحية الحاسوبية، تتضمن حل مسألة المجموع الجزئي دون الاستفادة من تفكيك الاعداد الى أجزاء فردية، الامر الذي يجعل التعامل مع الاعداد الكبيرة واستخلاص بنيتها بكفاءة امرا غير عملي. يركز هذا البحث على الاعداد الغريبة التي تكون على شكل ($\boldsymbol{n = 2^k p\, q}$) مع توسيع الدراسة لتشمل الاعداد الغريبة من الشكل ($\boldsymbol{n = 2^k p q_1 q_2}$) كما يتم اشتقاق صيغ صريحة لكل من دالة اولير ($\boldsymbol{\varphi(n)}$) ودالة مجموع القواسم $\boldsymbol{\sigma(n)}$ودراسة التغيرات التي تطرأ عليهما بصيغتهما الضربية ،

وتحويل هذه الصيغ الى تقديرات تقاربية تتضمن حدودا للخطأ. ومن أبرز النتائج التوصل الى كمية ترشيح ((**Filtering Quantity**) تعد مؤشرا اوليا على غرابة العدد. واستنادا الى النتائج العددية، يتم التحقق من دقة التقريبات وبيان فاعلية المرشح المقترح، بما يوفر إطارا يمكن تطويره وتوسيع نطاق تطبيقه للكشف عن الاعداد الغريبة بطريقة قابلة للتوسيع وذات كفاءه حسابية.

الكلمات المفتاحية: دالة اولير، دالة مجموع القواسم، الأعداد الغريبة، الأعداد غير الكاملة.

1. Introduction

The number theory class of weird numbers is an abundant, but at the same time non-semiperfect class of numbers with interesting distinctive properties. A positive integer n is abundant if the sum of its divisors other than n is greater than $\sigma(n) > 2n$, Where $\sigma(n)$ denotes the divisor sum function.

The other new restriction here creates a significant combination condition, with caused weed-like, the numbers differ significantly from the others abundant. Due to this dual nature, weird numbers lie at the intersection of multiplicative and additive number theory. On the one hand, their definition involves divisor functions and prime factorization, which are inherently multiplicative in nature. The need to be non-semi perfect, however, leads to an additive constraint associated with subset-sum problems. Usually, weirdo relies on the following two conditions: abundance and non-semi perfectness. It is easy to check whether the divisor function is abundant, while checking a non-semi perfectness of a divisor function is not efficient, since all subsets of the set of proper divisors need to be checked. It is a sigma verification problem which is known to be complex conspiratorially, and the computational cost of the problem increases rapidly as the number of its divisors increases.

Classical methods are thus inapplicable when the size of the number is large, especially when it has many prime factors. The limitation discussed here has motivated recent research into other techniques, which avoid the need for extensive computation and make use of the structural properties of integers. The function that is multiplicative, like Euler's function $\varphi(n)$ and the sigma function $\sigma(n)$ is a useful framework within which analysis of this problem can be done. A function, Euler's $\varphi(n)$, which is central to number theory, is used to count the number of numbers that are relatively prime to n. Likewise, the sigma function $\sigma(n)$ expresses the additive nature of the divisors and helps identify numbers as deficient, perfect or plentiful. We propose to analyze these weird numbers through the interaction between the two functions instead of analyzing them individually. Here we use normalized quantities to reach this aim,

$$T(n) = \frac{\varphi(n)}{n}, \quad S(n) = \frac{\sigma(n)}{n} \tag{1}$$

Which allow meaningful comparison between integers of different magnitudes.

These can be normalized functions that represent the relative arithmetic behavior of the integer, thus removing scaling effects, and they are especially useful in some form for structural analysis. The main object of this study is the set of integers of the type

$n = 2^k p q$, where p and q are distinct prime numbers and $k \geq 1$

These numbers are of special concern since many of the known weird numbers come from this class, and this yields a set of manageable structure on which to obtain explicit analytical formulas. We extend this analysis further to integers of the form $n = 2^k p q_1 q_2$. In order to investigate the effect of additional prime factors on the behavior of the proposed framework. By exploiting the multiplicative properties of $\varphi(n)$ and $\sigma(n)$, we derive explicit closed-form expressions and asymptotic representations for the normalized functions $T(n)$ and $S(n)$. A key contribution of this work is the identification of a direct

analytical relationship between $T(n)$ and $S(n)$, which leads to the introduction of a new filtering quantity,

$$Wz(n) = S(n) - 4T(n) \quad (2)$$

This number can be used here as a preliminary indicator of weirdness, being a measure of the structure of the distinction between the weird and non-weird numbers in a numerical class. The proposed filter is not required or sufficient for weirdness, but essentially acts like a fast and cheap proxy filter to cut back on the search space by applying more sophisticated filters at a later stage. The numerical experiments presented below have been performed to illustrate the effectiveness of this approach, and the results indicate that with large prime factors, high accuracy can be obtained for the approximation formulas. Moreover, the suggested framework develops a theoretical understanding of how weird numbers may take certain structural patterns, as opposed to just jumping right to computation. The present study not only connects number theory with computing, but also provides a close theoretical analysis of studying weird numbers and suggests future avenues for study.

2. Research Problem

Although much of the study of arithmetic functions and structures of divisors in number theory has been done analytically, the weird numbers are quite poorly understood analytically. The main challenge that can be expressed is the dual nature of them, that is, they are not only rich but also non-semiperfect. Although the abundance condition can be checked efficiently by solving a subset-sum problem over the set of proper divisors, the non-semiperfect condition does not have such an efficient check. It is a combinatorial problem which turns computationally infeasible on large integers because of an exponential change in the number of subsets.

Current strategies to determine weird numbers are largely based on exhaustive computational techniques, and little provision is made for developing analytical criteria or structural indicators which can be used to anticipate weird numbers. Specifically, scalable methods that minimize the computational problem and still maintain accuracy are lacking. Thus, the core research question that is the subject matter of the present research is: How can the use of analytical and structural properties of arithmetic functions allow the characterization and detection of weird numbers in a fashion that reduces or avoids exhaustive subset-sum computations?

3. Research Objectives

The main objective of this research is to develop an analytical framework for studying weird numbers using normalized arithmetic functions and multiplicative structures. More specifically, this study aims to:

1. **Develop normalized representations** of Euler's function and the divisor sum function to enable meaningful comparison between integers of different magnitudes.

$$T(n) = \frac{\varphi(n)}{n} \quad ,S(n) = \frac{\sigma(n)}{n} \quad (3)$$

2. Derive explicit and asymptotic formulas for $\varphi(n)$ and $\sigma(n)$ for integers of the form

$$n = 2^k\, p\, q \quad and \quad n = 2^k\, p\, q_1 q_2 \quad (4)$$

3. Analyze the interaction between $T(n)$ and $S(n)$ in order to uncover structural patterns associated with weird numbers.
4. Introduce a new filtering quantity, called the Zwain Weirdness Filter

$W_z(n) = S(n) - 4T(n)$ and evaluate its effectiveness as a preliminary indicator of weirdness.

5. Reduce computational complexity by providing a screening method that limits the need for exhaustive subset-sum verification.
6. Validate the proposed framework through numerical experiments and error analysis of the derived approximations.

4. Research Novelty and Significance

The novelty of this work is the direct relationship between the arithmetic properties of Euler's function and the sigma function and their structural characterization of weird numbers. Instead of using classical methods focusing heavily on exhaustive subset-sum testing, this study suggests a two-density method where the analytical and asymptotic properties of these functions are first used to "filter out" the integers of high weirdness potential. This will make the calculations easier and give an insight into the nature of the weird numbers. Furthermore, the establishment of the controlled approximation formulae with explicit error bounds gives grounds for developing scalable algorithms that can operate over wide ranges of numerical values. This chapter not only deepens the theoretical knowledge of weird numbers, but it also initiates new ways of faster association and broader classifications for the analysis of multiplicative numbers.

5. Contribution of the Research

In this work, the study of weird numbers is improved by using an analytical framework by normalized arithmetic functions. The main contributions of this work are listed below. First, closed forms for the Euler totient and for divisor sum functions were found for integers with one special structural form investigated in this study. The equations give the ability to compute these functions without having to select the divisors, directly from the structure of the integer in terms of its arithmetic. Secondly, they added what they called normalized quantities based on Euler and Sigma functions. These would enable the numerical comparison of integers, no matter how large their values, and they would help to identify structural patterns related to weird numbers. Thirdly, a new filtering amount related to the normalized functions was found and created. This filter offers a computationally efficient preliminary screening filter that can be used to filter out candidate weird numbers for which the computational requirement is many times less than tests based on subset-sums. Finally, the filtering methodology presented here in the numerical experiments, indicates that the set of proposed values for the filter has a relatively stable numerical range where the weird numbers are located, pointing to the potential ability of the methodology to reduce the number of cases that should be considered when investigating weird numbers.

6. Basic Definitions and Concepts Introduction

Definition 6 .1: [12]

The divisor function $\boldsymbol{d}(\boldsymbol{n})$ is defined as the number of divisors of n , such that $\boldsymbol{n} \in \boldsymbol{Z}^+$

$$\text{i.e. } \boldsymbol{d}(\boldsymbol{n}) = \sum_{d/n} \mathbf{1} \tag{5}$$

Example 6.1: $n = 12$

$$(12) = \sum_{d/12} \mathbf{1} = 6 \tag{6}$$

Definition 6. 2: [12]

The Euler function φ (n) is the number of positives such K relatively with .

$$\boldsymbol{i.e.}\ \varphi(n) = \sum_{k=1}^{n} 1 \quad , \mathbf{1} \leq \boldsymbol{k} \leq \boldsymbol{n} \quad ,(\boldsymbol{k}, \boldsymbol{n}) = \mathbf{1}. \qquad (6.2)$$

Example 6 .2: $n = 12$

$\varphi(12) = \sum_{k=1}^{n=12} 1 \quad \mathbf{1} \leq \boldsymbol{k} \leq 12 \quad ,(\boldsymbol{k}, 12) = \mathbf{1}$

$\varphi(12) = 4$

Definition 6.3: [12]

The sum of divisors function, denoted by $\boldsymbol{\sigma}(\boldsymbol{n})$, is the sum of all positive divisors

Note that we can express it as $\sigma(n) = \sum_{d/n} \boldsymbol{d}$.

Example 6.3: $\boldsymbol{n} = 12$

$\boldsymbol{\sigma}(12) = \sum_{\boldsymbol{d}/\boldsymbol{n}} \boldsymbol{d}$

$\boldsymbol{\sigma}(12) = 1+2+3+4+6+12 = 28$

$\sigma(12) = 28$

Definition 6 .4: [12]

An arithmetic function is said to be a multiplicative function if

$\boldsymbol{f}(\boldsymbol{a}\,\boldsymbol{b}) = \boldsymbol{f}(\boldsymbol{a}).\,\boldsymbol{f}(\boldsymbol{b})$ whenever $(a, b) = \mathbf{1}$.

Example 6.4:

The Euler function φ (n) an arithmetical function

$\varphi(12) = \varphi(3.4) = \varphi(3)\varphi(4) = 2.2 = 4$

Definition 6.5 [12]

Let n is a natural number if the sum of the proper divisors of except itself is greater than $\boldsymbol{n}$, $\boldsymbol{n}$ is called an abundant number.

Example 6.5:

Let $n = 12$

The proper divisors of 12 are 1,2,3,4,6

$1+2+3+4+6 = 16 > 12$ Then 12 is an abundant number

Definition 6.6 [12]

Let is a natural number if the sum of the proper divisors of except itself is less then $\boldsymbol{n}$, $\boldsymbol{n}$ is called deficient number.

Example 6.6

Let n = 8. The proper divisors of 8 are 1, 2, 4

1+2+4= 7 $<$ 8 Then 8 is a deficient number.

Definition 6.7[12]

Let n is a natural number the sum of some or all its distinct proper divisors of except itself, $\boldsymbol{n}$ is called semiperfect number.

Example 6.7

Let n= 12, and some of the proper divisors of 12 are 2,4,6 then 2+4+6 =12

Definition 6.8 [12]

Let $\mathbf{n}$ is a natural number if any abundant number but not semiperfect $\boldsymbol{n}$ is called weird number.

Example 6.8

Let $\boldsymbol{n} = \mathbf{70}$. Proper divisors of 70 are 1 , 2 , 5 , 7 , 10 , 14 , 35

1+2+5+7+10=35= 74 > 70, but for any subset of its divisors, they don't equal 70

Definition 6.9: [10 -12]

Theoretical Error Bound

Suppose a quantity admits an approximation of the form

$$\boldsymbol{x} = \boldsymbol{f}\ (\boldsymbol{n}) + \mathrm{O}\ (\boldsymbol{g}\ (\ \boldsymbol{n})) \qquad (7)$$

The term O ($\boldsymbol{g}$ ($\boldsymbol{n}$)) represents a theoretical bound on the order of the approximation error.

It indicates that the error does not exceed a constant multiple of $\boldsymbol{g}$ ($\boldsymbol{n}$) for sufficiently large parameters, and thus describes the asymptotic behavior of the approximation rather than its exact numerical deviation

Definition 6.10: [9] (Relative Error)

Let x denote the exact value of a quantity and $\tilde{x}$ approximation. The relative error of the approximation is defined by $\varepsilon = \frac{|x - \tilde{x}|}{|x|}$

This quantity measures the deviation of the approximation compared with the magnitude of the exact value, and therefore provides a practical indicator of numerical accuracy.

7. previous Studies on weird Numbers

There have been a number of studies of weird numbers and some of their properties. A number is called a weird number if it is an abundant number and is not the sum of any subset of its proper divisors. They have been the subject of great interest in number theory due to their peculiar construction. One of the earliest theoretical studies of weird numbers was presented by Kriickeberg (1970). In his work, he was able to prove that there are infinitely many weird numbers; he also found many weird numbers that could be built from numbers of the form $\boldsymbol{n} = \mathbf{2}^{\boldsymbol{k}}\ \boldsymbol{pq}$ where p and q are distinct odd primes and k ≥ $\mathbf{1}$. This result was the basis of future studies.

Erdos (1973) investigated lots of numbers and other divisor problems. The work he did gave theoretical ideas into abundant numbers, which are well related to weird numbers, as every weird number

is an abundant number. Benko (1974) studied the structural characteristics of the "weird numbers" and gave a number of numerical examples. By resolving the relationship between abundant numbers and weird numbers, his study brought the emphasis that every weird number is abundant, but the converse is not true. Benkoski and Erdos (1974) studied primitive abundant numbers and related divisor problems. Their work contributed to understanding the structural conditions under which abundant numbers may exhibit special properties such as weirdness.

Hagis and McDaniel (1974) analyzed properties of divisor functions and their applications to special classes of integers. Their work helped provide theoretical tools that are useful in the study of weird numbers. Guy and Selfridge (1975) investigated several unusual properties of integers related to divisor sums. Their work contributed the theoretical framework that supports the study of weird numbers. Norton (1976) examined number theoretic properties related to divisor functions and abundant numbers, providing additional theoretical insights relevant to the structure of weird numbers. Melfi (1999) studied primitive weird numbers and proved that infinitely many primitive weird numbers exist. His work showed that numbers of the form $\boldsymbol{2^k\, p\, q}$ appear frequently in the structure of primitive weird numbers.

Guy (2004) discussed weird numbers in his well-known book Unsolved Problems, especially the question of whether an odd weird number exists. Up to now, no odd weird number has been discovered. Fang and Chen (2015) carried out computational investigations on weird numbers and extended the known numerical data related to these numbers using computational techniques. Amato et al. (2018) developed computational algorithms to detect weird numbers and expanded tables of weird numbers to larger numerical ranges using modern computational methods.

8. Euler's Function Formula

Theorem 8.1 (Euler's totient formula)

For all weird number with the form $2^k\ p\ .\ q$ where p and q are distinct odd primes and $k \geq 1$. Then

$$\varphi(n) = 2^{k-1}(p\ -\ 1)(q\ -\ 1). \tag{8}$$

Proof. The numbers $2^k, p$ and q are pairwise coprime and φ is multiplicative, so $\varphi(n) = \varphi(2^k)\varphi(p)\varphi(q)$. Moreover $\varphi(2^k) = 2^k - 2^{k-1} = 2^{k-1}$, and

$\varphi(r)\ =\ r\ -\ 1$ for every prime r

then $\varphi(n)\ = 2^{k-1}(p\ -\ 1)(q\ -\ 1)$.

Theorem 8.2 (Sigma function formula)

For all n weird numbers with the form $2^k\ .\ q$, where p and q are distinct odd primes and $k \geq 1$. Then

$$\sigma(n) = (2^{k+1} - 1)(p + 1)(q + 1) \tag{9}$$

Proof. The numbers 2^k, p and q are pairwise coprime, and σ is multiplicative, so $\sigma(n) = \sigma(2^k)\ \sigma(p)\ \sigma(q)$. The divisors of 2^k are 1, 2, 2^2, …, 2^k, hence by the geometric series $\sigma(2^k) = 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$. For a prime r, the only divisors are 1 and r, so $\sigma(r) = r + 1$. Then

$$\sigma(n) =\ (\ 2^{k+1}\ -\ 1)\ \ (\ p + 1)\ \ (q + 1) \tag{10}$$

From the derivation of the simple Classical formulas for Euler's and Sigma functions when the number is Weird and of the form (. $2^k p.\ q$). We proceed to derive new formulas based on those two functions. The aim is to reduce the computational complexity normally required by Euler and sigma functions, which typically demand full factorization and enumeration of many divisors. New formulas have been derived that depend only on the Structure $2^k\ .\ p.\ q$ of the number. Using these new formulas, a researcher can efficiently filter a number and determine whether it is weird or not, without needing to calculate all of its divisors.

Example 8.3 (Euler's function)

1) n = 836 = $2^2 \cdot 11 \cdot 19$ (k = 2): $\varphi(836) = 2^1 \cdot 10 \cdot 18 = 360$.

2) n = 70 = $2 \cdot 5 \cdot 7$ (k = 1): $\varphi(70) = 2^0 \cdot 4 \cdot 6 = 24$.

3) n = 7192 = $2^3 \cdot 29 \cdot 31$ (k = 3): $\varphi(7192) = 2^2 \cdot 28 \cdot 30 = 3360$.

Example 8.4 (Sigma function)

1) n = 836 = $2^2 \cdot 11 \cdot 19$ (k = 2): σ (836) = $(2^3 - 1) \cdot 12 \cdot 20 = 1680$.

2) n = 70 = $2 \cdot 5 \cdot 7$ (k = 1): σ(70) = $(2^2 - 1) \cdot 6 \cdot 8 = 144$.

3) n = 7192 = $2^3 \cdot 29 \cdot 31$ (k = 3): σ(7192) = $(2^4 - 1) \cdot 30 \cdot 32 = 14400$

Proposition 8.5 (Representation of Euler's Ratio)

Let $n = 2^k \cdot p \cdot q$, where p and q are distinct odd primes and k ≥ 1, and let

T(n) = $\frac{\varphi(n)}{n}$. Then

$$T(n) = \frac{1}{2}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) = \frac{1}{2}\left[1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{p.q}\right] \tag{11}$$

Consequently:

$$T(n) = \frac{1}{2}\ \left(1 - \frac{1}{p} - \frac{1}{q}\right) + E, \text{ where } E = \frac{1}{2p.q}, \text{ so } E = O\left(\frac{1}{p.q}\right) \tag{12}$$

$$T(n) = \frac{1}{2} + E', \text{ where } |E'| \le \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right), \text{ so } E' = O\left(\frac{1}{p} + \frac{1}{q}\right) \tag{13}$$

Proof. By Theorem 8.1,

$$\frac{\varphi(n)}{n} = \frac{2^{k-1}(p-1)(q-1)}{2^k .p.q}$$

$$= \frac{1}{2}\ \left(1 - \frac{1}{p}\ \right)\ \left(1 - \frac{1}{q}\ \right)$$

Expanding the multiplicative $\frac{\varphi(n)}{n} = \frac{1}{2}\ \left(1 - \frac{1}{p}\ \right)\ \left(1 - \frac{1}{q}\ \right)$

$$T(n) = \frac{1}{2}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) = \frac{1}{2}\left[1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{p.q}\right].$$

Now, the value of $\frac{1}{p.q}$ Approximated it.

So $\frac{\varphi(n)}{n} \approx \frac{1}{2} - \frac{1}{2p} - \frac{1}{2p} + o\left(\frac{1}{pq}\right)$

$|E\varphi| = \frac{1}{2.p.q}$ error term or

$\frac{\varphi(n)}{n} = \frac{1}{2} + o\left(\frac{1}{p} + \frac{1}{q}\right)$. With error term $|E'| = \left(\frac{1}{p} + \frac{1}{q}\right)$

Remark. T(n) does not depend on k. Approximation (i) has the smaller error and is the one used in this study to connect T(n) with S(n). Approximation (ii) is cruder but cheaper, and is suitable only as a first, rapid screening. In the first approximation, we neglected the smaller term $\frac{1}{p.q}$ While in the second approximation, we neglected the term $\frac{1}{p} + \frac{1}{q}$. However, in this study, we relied on the first approximation to establish a more accurate connection with the function. Accordingly, this yields the final approximation formula for Euler's function,

$$T(n) = \frac{\varphi(n)}{n} \cong \frac{1}{2}\left(1 - \frac{1}{p} - \frac{1}{q}\right) + o\left(\frac{1}{p.q}\right) \tag{14}$$

8.6 Transformation of $\frac{\sigma(n)}{n}$ into an asymptotic form

Starting from the exact multiplicative formula of the sigma function

$$\frac{\sigma(n)}{n} = \prod \frac{p^{k+1}-1}{p^k(p-1)} \tag{15}$$

We consider integers of the form $= 2^k\ p.\ q$. For such integers, the formula can be rewritten by separating the dominant term from the smaller corrective contributions. For the odd primes p and q, their contributions to the product can be expressed as terms of quantities of order $\frac{1}{p}$ and $\frac{1}{q}$, whereas the contribution of the power of 2 appears as a small term of order 2^{-k}. Consequently, the ratio may be written in the form $\frac{\sigma(n)}{n} = 2 + \frac{2}{p} + \frac{2}{q} + O\left(2^{-k} + \frac{1}{p.q}\right)$ Where the dominant part represents the principal value of the ratio, while the remaining terms reflect the influence of secondary prime factors and powers. It should be emphasized that this expression does not represent a numerical approximation of the function, but rather an analytical rewriting of the exact formula. It preserves the full arithmetic structure of the divisor function while highlighting the dominant contributions, which facilitates theoretical analysis, comparison with Euler's ratio $T(n)$, and its use in screening procedures, as will be shown in the following.

Proposition 8.7 Representation of sigma Ratio

Let $n = 2^k.\ p\ .q$ with $k \geq 1$, p, q distinct primes and $\sigma(n)$ is multiplicative function then

$$\frac{\sigma(n)}{n} = \sigma(2^k)\,(\sigma(p)\,\sigma(q))$$

$$= (2^{k+1} - 1)\,(p+1)\,(q+1)$$

Divided by $n = 2^k\ .p\ .q$ and simplifying gives

$$\frac{\sigma(n)}{n} = (2 - 2^{-k})\left(1 + \frac{1}{p}\right)\left(1 + \frac{1}{q}\right)$$

$$= 2\left(1 + \frac{1}{p} + \frac{1}{p} + \frac{1}{pq}\right) - 2^{-k}\left(1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{pq}\right)$$

$$S(n) = \frac{\sigma(n)}{n} = 2\left(1 + \frac{1}{p} + \frac{1}{p} + \frac{1}{pq}\right) - 2^{-k}\left(1 + \frac{1}{p}\right)\left(1 + \frac{1}{q}\right)$$

$$S(n) = \frac{\sigma(n)}{n} \cong 2 + \frac{2}{p} + \frac{2}{q} + O\left(2^{-k} + \frac{1}{p\,q}\right) \quad (16)$$

As for the sigma function, we con sidereal the approximation to coincide with exact formula, with zero error. However, it may be written in this form separating the dominant term from the full arithmetic structure and allows both theoretical analyses. Therefore, as we noted above the proof of this representation is straight forward, as it follows directly from the classical formula upon division by n.

Corollary 8.9 (Approximate linear relation between $T(n)$ and $S(n)$)

Let $n = 2^k \cdot p \cdot q$ and define $S(n) = \frac{\sigma(n)}{n}$, $T(n) = \frac{\sigma(n)}{n}$ then

$S(n) = 4 - 4\,T(n) + O\left(2^{-k} + \frac{1}{p\,q}\right)$.

In particular, S(n) is approximately decreasing linear function of T(n)

$$T(n) = \frac{1}{2} - \frac{1}{2p} - \frac{1}{2p} + O\left(\frac{1}{p.q}\right)$$

$$4\,T(n) = 2 - \frac{2}{p} - \frac{2}{q} + O\left(\frac{1}{p.q}\right)$$

So,

$$S(n) = 2\frac{2}{p} + \frac{2}{q} + O\left(2^{-k} + \frac{1}{p\,q}\right)$$

Comparing the last two displays Yield:

$$S(n) = 4 - 4\,T(n) + O\left(2^{-k} + \frac{1}{p\,q}\right) \quad (17)$$

This relationship forms the theoretical basis of the proposed Zwain Weirdness Filter.

Theorem 8.10

Let n positive integer number and $S(n) = \frac{\sigma(n)}{n}$ and $T(n) = \frac{\varphi(n)}{n}$

Then

$$W_z(n) = S(n) - 4T(n) + O\left(\frac{1}{2.p.q}\right) \quad (18)$$

Where the approximation error of $T(n)$ is of order $O\left(\frac{1}{2.p.q}\right)$

Definition 8.11(Zwain Weirdness Filter).

Let $T(n) = \frac{\varphi(n)}{n}$ and $S(n) = \frac{\sigma(n)}{n}$.

We define the Zwain Weirdness Filter as

$$W_Z(n) = S(n) - 4\,T(n) \quad (19)$$

9. Numerical Evaluation of the Approximation

The proposed quantity depends solely on the numerical structure of the integer n of the form, and it does not require divisor enumeration or the construction of subset sums of divisors. Consequently, it can be computed with a computational cost that is significantly lower than that required for the semi perfectness test in practical applications, this quantity can therefore be used to generate a list of candidate weird numbers within the class $2^k.\,p.\,q$. Since the properties of abundance and non-semi perfectness are

reflected in the behavior of the normalized arithmetic functions (1) and (2) the proposed filter W_z (n) captures this structural distinction. In particular, it manifests as a characteristic numerical interval that tends to contain weird numbers while separating them from non-weird integers within the same numerical family.

This observation provides empirical and analytical support for the suitability of (W_z (n)) as a preliminary screening criterion within the scope of this study. It should be emphasized that the proposed filter is neither a necessary nor a sufficient condition for weirdness. Rather, it serves as an efficient preliminary screening indicator derived from numerical and statistical observations of integers n of the form (2^k. p, q), allowing the search space for weird numbers to be significantly reduced before applying more computationally intensive tests.

10. Accuracy of the Approximation Formula Based on Numerical Results

7192 = 2^3.29.31
Exact: T(n) = 420/899 ≈ 0.4672
Approx: T(n) ≈ 0.4661
806 = 2.13.31
Exact: T(n) = 180/403 ≈ 0.4467
Approx: T(n) ≈ 0.4454
986 = 2.17.29
Exact: T(n) = 224/493 ≈ 0.4544
Approx: T(n) ≈ 0.4534

The numerical computations for the selected integers show that the approximation for the ratio $T(n) = \frac{\varphi(n)}{n}$ given by the relation in Equ. (20) bellow achieves a high degree of accuracy when the integer is of the form 2^kpq or when it is effectively governed by two large prime factors.

$$T(n) \approx \frac{1}{2}\left(1 - \frac{1}{p} - \frac{1}{q}\right) + o\left(\frac{1}{p.q}\right) \qquad (20)$$

In the computed examples, such as 806 = 2.13.31 and 986 = 2.17.29, the difference between the exact value and the approximate value was very small, with the relative error on the order of a few thousandths. This confirms that the formula provides a good estimate of the ratio $T(n)$ in such cases. The calculations for the integer 7192 =2^3.29.31 also show that the presence of powers of 2 does not affect the accuracy of the approximation, since Euler's formula depends only on the distinct prime factors rather than their powers. Consequently, the relative error remains small in this case as well.

These results indicate that the theoretical error term of order O ($\frac{1}{p\,q}$) provides a suitable estimate for the magnitude of the actual deviation whenever the integer is structurally close to the model assumed in the approximation. Therefore, the proposed formula can be adopted as a fast computational tool for estimating the ratio $T(n)$ and for filtering integers in numerical investigations, while exact computation may be used later when final verification is required.

11 Analytical Motivation for the Weirdness Filter

After deriving the classical closed-form expressions for the Euler's function and the divisor sum function when n is weird number of the form ($2^k\ p\,.q$), both functions were written explicitly in terms of the numerical structure of the integer. This formulation avoids the need for divisor enumeration or repeated exhaustive factorization. To facilitate numerical comparison between integers of different magnitudes, the two functions were normalized by dividing each of them by n. accordingly, the normalized quantities $T(n)$ and $S(n)$ were introduced.

Since weird numbers are abundant but not semiperfect, the behavior of the normalized quantities $T(n)$ and $S(n)$ for weird numbers exhibits a structural pattern that differs from that observed for non-weird integers within the same numerical class. This observation motivates the idea of combining these two quantities into a single numerical measure that can act as a preliminary filtering criterion. Such a criterion allows the rapid identification of integers that are likely to be weird, with a computational cost significantly lower than direct methods based on testing subsets of divisors. Based on this idea, a linear combination was constructed which leads to a practical and computationally efficient filtering indicator.

Theorem 11.1

For all n weird number with the form $2^k\ p.\ q_1, q_2$ where p and q_1, q_2 are distinct odd primes and k ≥ 1. Then

$$\varphi(n) = 2^{k-1}\ (p-1)\ (q_1-1)\ (q_2-1) \tag{21}$$

Proof. The numbers 2^k , p, q_1, q_2 are pairwise coprime and φ is multiplicative, so $\varphi(n) = \varphi(2^k)\varphi(p)\varphi(q_1)\ \varphi(q_2)$

Moreover $\varphi(2^k) = 2^k - 2^{k-1} = 2^{k-1}$, and $\varphi(r) = r - 1$ for every prime then

$\varphi(n) = 2^{k-1}(p-1)(q1-1)\ (q2-1)$

Theorem 11.2

Let $n = 2^k\, p\, q_1\, q_2$ with $k \geq 1$ and p, q_1, q_2 pairwise distinct odd primes. Then

$$\frac{\varphi(n)}{n} = \frac{1}{2}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)$$

$$= \frac{1}{2}\left[1-\left(\frac{1}{p}+\frac{1}{q_1}+\frac{1}{q_2}\right)+\left(\frac{1}{pq_1}+\frac{1}{pq_2}+\frac{1}{q_1q_2}\right)-\frac{1}{pq_1q_2}\right].$$

Consequently, for large primes,

$$\frac{\varphi(n)}{n} \approx \frac{1}{2}\left(1-\frac{1}{p}-\frac{1}{q_1}-\frac{1}{q_2}\right), \tag{22}$$

with error term

$$E = \frac{1}{2}\left(\frac{1}{pq_1}+\frac{1}{pq_2}+\frac{1}{q_1\,q_2}-\frac{1}{pq_1q_2}\right). \tag{23}$$

If $m = \min\{p, q_1, q_2\}$, then $0 < E < \frac{3}{2m^2}$.

Proof. The result follows from the multiplicativity of Euler's totient function, , where the product runs over the distinct primes dividing , together with direct expansion of the product$\varphi(n) = n\prod_{r|n}\left(1-\frac{1}{r}\right) rn.$

Theorem 11.3 Let $n = 2^k\, p\, q_1\, q_2$ with $k \geq 1$ and p, q_1, q_2 pairwise distinct odd primes. Since σ is multiplicative and $\sigma(r^a) = \frac{r^{a+1}-1}{r-1}$,

So $\sigma(2^k) = \frac{2^{k+1}-1}{2-1} = \ 2^{k+1}-1$

$\sigma(n) = \sigma(2^k)\sigma(p)\sigma(q_1)\sigma(q_2)$,

$$\sigma\ (n) = (\ 2^{k+1}\ -\ 1\ )\ (\ p\ +\ 1\ )\ (\ q_1\ +\ 1\ )\ (\ q_2\ +\ 1) \tag{24}$$

Theorem 11.4. Let $n = 2^k\, p\, q_1\, q_2$ with $k \geq 1$ and p, q_1, q_2 pairwise distinct odd primes. Since σ is multiplicative and $\sigma(r^a) = \frac{r^{a+1}-1}{r-1}$,

$$\sigma(n) = (2^{k+1} - 1)(p + 1)(q_1 + 1)(q_2 + 1),$$

and hence

$$\frac{\sigma(n)}{n} = \frac{2^{k+1}-1}{2^k} \cdot \frac{p+1}{p} \cdot \frac{q_1+1}{q_1} \cdot \frac{q_2+1}{q_2} = \left(2 - \frac{1}{2^k}\right)\left(1 + \frac{1}{p}\right)\left(1 + \frac{1}{q_1}\right)\left(1 + \frac{1}{q_2}\right).$$

Moreover, to first order in the reciprocals of the primes,

$$\frac{\sigma(n)}{n} \approx \left(2 - \frac{1}{2^k}\right)\left(1 + \frac{1}{p} + \frac{1}{q_1} + \frac{1}{q_2}\right).$$

$$\frac{\sigma(n)}{n} \approx \left(2 - \frac{1}{2^k}\right)\left(1 + \frac{1}{p}\right)\left(1 + \frac{1}{q_1}\right)\left(1 + \frac{1}{q_2}\right)$$

Expanding to first order gives.

$$\frac{\sigma(n)}{n} \approx 2 + 2\left(\frac{1}{p} + \frac{1}{q_1} + \frac{1}{q_2}\right) - \frac{1}{2^k} \qquad (25)$$

Corollary 11 .5 Relation between Euler's function and the sigma function

For $n = 2^k p\, q_1 q_2$ the divisor - sum ratio satisfied

$$\frac{\sigma(n)}{n} = \frac{2^{k+1}-1}{2^k} \cdot \frac{p+1}{p} \cdot \frac{q_1+1}{q_1} \cdot \frac{q_2+1}{q_2}$$

For the large primes $p, q_1 q_2$, the first -order approximation is

$$\frac{\sigma(n)}{n} \approx \left(2 - \frac{1}{2^k}\right)\left(1 + \frac{1}{p}\right)\left(1 + \frac{1}{q_1}\right)\left(1 + \frac{1}{q_2}\right)$$

Expanding to first order gives.

$$\frac{\sigma(n)}{n} \approx 2 + 2\left(\frac{1}{p} + \frac{1}{q_1} + \frac{1}{q_2}\right) - \frac{1}{2^k}$$

On the other hand, from the first -order approximation of Euler's ratio we obtain

$$\frac{\varphi(n)}{n} \approx \frac{1}{2}\left(1 - \frac{1}{p} + \frac{1}{q_1} + \frac{1}{q_2}\right).$$

We obtain

$$\frac{1}{p} + \frac{1}{q_1} + \frac{1}{q_2} \approx 1 - 2\frac{\varphi(n)}{n}$$

Substituting this expression into the first-order approximation of the sigma function ratio yields.

$$\frac{\sigma(n)}{n} \approx 4 - 4\,\frac{\varphi(n)}{n} - \frac{1}{2^k} \quad \text{i.e } S(n) \approx 4 - 4T(n) \qquad (26)$$

This formula establishes a direct analytical relationship between Euler's function and the divisor-sum function.

Proposition: 11.6

Weirdness Indicator for $n = 2^k . p . q_1 . q_2$ let $n = 2^k . p . q_1 . q_2$ where $k \geq 1$ and $. p . q_1 . q_2$ are distinct odd primes

Define

$$S(n) = \frac{\sigma(n)}{n},$$

$$T(n) = \frac{\varphi(n)}{n}$$

Introduce the Zwain Weirdness Filter

$$Wz(n) = S(n) - 4\,T(n)$$

so ,

$$Wz(n) = \left(2 - \frac{1}{2^k}\right) \prod_{i=1}^{3} \left(1 + \frac{1}{p_i}\right) - 2 \prod_{i=1}^{3} \left(1 - \frac{1}{p_i}\right),\ p_i \in \{p, q_1, q_2\}$$

Moreover, for large primes.

$$Wz(n) \approx 4\left(\frac{1}{p} + \frac{1}{q_1} + \frac{1}{q_2}\right) - \frac{1}{2^k} \quad (27)$$

Consequently, an integer as a form $n = 2^k . p . q_1 . q_2$ may be seen as a potential candidate for weirdness whenever it satisfies the heuristic condition.

$\frac{2^{k+1}-1}{2^k} \prod_{i=1}^{3} (1 - \frac{1}{p_i}) > \prod_{i=1}^{3}(1+\frac{1}{p_i})$, where $p_i \in \{ p , q_1, q_2\}$

This serves as preliminary screening criteria derived from Zwain Weirdness Filter and is intended to reduce the search space for candidate weird numbers, as illustrated in bellow Tables .

Table 1: Weird $n = 2^k p\, q$

Type of n	N	k	p	q	φ(n)	T(n)	| E φ|	σ (n)	S(n)	| Eσ|	Wz(n)
Weird	836	2	11	19	360	0.43062	0.002392	1680	2.00957	0	0.28708
Weird	7192	3	29	31	3360	0.46719	0.000556	14400	2.00222	0	0.13348
Weird	7912	3	23	43	3696	0.46714	0.000506	15840	2.00202	0	0.13347
Weird	9272	3	19	61	4320	0.46592	0.000431	18600	2.00604	0	0.14236
Weird	10792	3	19	71	5040	0.46701	0.000371	21600	2.00148	0	0.13343
Weird	22810	1	5	2281	9120	0.39982	0.000044	41076	1.80079	0	0.20149
Weird	23090	1	5	2309	9232	0.39983	0.000043	41580	1.80078	0	0.20147
Weird	23510	1	5	2351	9400	0.39983	0.000043	42336	1.80077	0	0.20145
Weird	24770	1	5	2477	9904	0.39984	0.000040	44604	1.80073	0	0.20137
Weird	27070	1	5	2707	10824	0.39985	0.000037	48744	1.80066	0	0.20126
Weird	27490	1	5	2749	10992	0.39985	0.000036	49500	1.80065	0	0.20124
Weird	27770	1	5	2777	11104	0.39986	0.000036	50004	1.80065	0	0.20122
Weird	28190	1	5	2819	11272	0.39986	0.000035	50760	1.80064	0	0.20121
Weird	28610	1	5	2861	11440	0.39986	0.000035	51516	1.80063	0	0.20119
Weird	29390	1	5	2939	11752	0.39986	0.000034	52920	1.80061	0	0.20116
Weird	30490	1	5	3049	12192	0.39987	0.000033	54900	1.80059	0	0.20112
Weird	31190	1	5	3119	12472	0.39987	0.000032	56160	1.80058	0	0.20109
Weird	33070	1	5	3307	13224	0.39988	0.000030	59544	1.80054	0	0.20103
Weird	34610	1	5	3461	13840	0.39988	0.000029	62316	1.80052	0	0.20098
Weird	35110	1	5	3511	14040	0.39989	0.000028	63216	1.80051	0	0.20097
Weird	35390	1	5	3539	14152	0.39989	0.000028	63720	1.80051	0	0.20096
Weird	35810	1	5	3581	14320	0.39989	0.000028	64476	1.8005	0	0.20095
Weird	36070	1	5	3607	14424	0.39989	0.000028	64944	1.8005	0	0.20094
Weird	36770	1	5	3677	14704	0.39989	0.000027	66204	1.80049	0	0.20092
Weird	37190	1	5	3719	14872	0.39989	0.000027	66960	1.80048	0	0.20091
Weird	37610	1	5	3761	15040	0.39989	0.000027	67716	1.80048	0	0.2009

Table 2: Non-weird $\boldsymbol{n = 2^k p\, q}$

Type of n	N	k	P	q	φ(n)	T(n)	| E φ|	σ (n)	S(n)	| Eσ|	Wz(n)
Non-weird	8	1	2	2	1	0.125000	0.125000	27	3.375000	0	2.875000
Non-weird	12	1	2	3	2	0.166667	0.083333	36	3.000000	0	2.333333
Non-weird	16	2	2	2	2	0.125000	0.125000	63	3.937500	0	3.437500
Non-weird	18	1	3	3	4	0.222222	0.055556	48	2.666667	0	1.777778

Non-weird	20	1	2	5	4	0.200000	0.050000	54	2.700000	0	1.900000
Non-weird	24	2	2	3	4	0.166667	0.083333	84	3.500000	0	2.833333
Non-weird	28	1	2	7	6	0.214286	0.035714	72	2.571429	0	1.714286
Non-weird	30	1	3	5	8	0.266667	0.033333	72	2.400000	0	1.333333
Non-weird	32	3	2	2	4	0.125000	0.125000	135	4.218750	0	3.718750
Non-weird	36	2	3	3	8	0.222222	0.055556	112	3.111111	0	2.222222
Non-weird	40	2	2	5	8	0.200000	0.050000	126	3.150000	0	2.350000
Non-weird	42	1	3	7	12	0.285714	0.023810	96	2.285714	0	1.142857
Non-weird	44	1	2	11	10	0.227273	0.022727	108	2.454545	0	1.545455
Non-weird	48	3	2	3	8	0.166667	0.083333	180	3.750000	0	3.083333
Non-weird	50	1	5	5	16	0.320000	0.020000	108	2.160000	0	0.880000
Non-weird	52	1	2	13	12	0.230769	0.019231	126	2.423077	0	1.500000
Non-weird	56	2	2	7	12	0.214286	0.035714	168	3.000000	0	2.142857
Non-weird	60	2	3	5	16	0.266667	0.033333	168	2.800000	0	1.733333
Non-weird	66	1	3	11	20	0.303030	0.015152	144	2.181818	0	0.969697
Non-weird	68	1	2	17	16	0.235294	0.014706	162	2.382353	0	1.441176
Non-weird	72	3	3	3	16	0.222222	0.055556	240	3.333333	0	2.444444
Non-weird	76	1	2	19	18	0.236842	0.013158	180	2.368421	0	1.421053
Non-weird	78	1	3	13	24	0.307692	0.012821	168	2.153846	0	0.923077
Non-weird	80	3	2	5	16	0.200000	0.050000	270	3.375000	0	2.575000
Non-weird	84	2	3	7	24	0.285714	0.023810	224	2.666667	0	1.523810
Non-weird	88	2	2	11	20	0.227273	0.022727	252	2.863636	0	1.954545
Non-weird	92	1	2	23	22	0.239130	0.010870	216	2.347826	0	1.391304
Non-weird	98	1	7	7	36	0.367347	0.010204	192	1.959184	1	0.489796
Non-weird	100	2	5	5	32	0.320000	0.020000	252	2.520000	2	1.240000
Non-weird	102	1	3	17	32	0.313725	0.009804	216	2.117647	3	0.862745
Non-weird	104	2	2	13	24	0.230769	0.019231	294	2.826923	4	1.903846
Non-weird	110	1	5	11	40	0.363636	0.009091	216	1.963636	5	0.509091
Non-weird	112	3	2	7	24	0.214286	0.035714	360	3.214286	6	2.357143
Non-weird	114	1	3	19	36	0.315789	0.008772	240	2.105263	7	0.842105
Non-weird	116	1	2	29	28	0.241379	0.008621	270	2.327586	8	1.362069
Non-weird	120	3	3	5	32	0.266667	0.033333	360	3.000000	9	1.933333
Non-weird	124	1	2	31	30	0.241935	0.008065	288	2.322581	10	1.354839
Non-weird	130	1	5	13	48	0.369231	0.007692	252	1.938462	11	0.461538
Non-weird	132	2	3	11	40	0.303030	0.015152	336	2.545455	12	1.333333
Non-weird	136	2	2	17	32	0.235294	0.014706	378	2.779412	13	1.838235
Type of n	**N**	**k**	**p**	**q**	**φ(n)**	**T(n)**	**| E φ|**	**σ (n)**	**S(n)**	**| Eσ|**	**Wz(n)**
Non-weird	140	2	5	7	48	0.342857	0.014286	336	2.400000	15	1.028571
Non-weird	148	1	2	37	36	0.243243	0.006757	342	2.310811	16	1.337838
Non-weird	152	2	2	19	36	0.236842	0.013158	420	2.763158	17	1.815789
Non-weird	154	1	7	11	60	0.389610	0.006494	288	1.870130	18	0.311688
Non-weird	156	2	3	13	48	0.307692	0.012821	392	2.512821	19	1.282051
Non-weird	164	1	2	41	40	0.243902	0.006098	378	2.304878	20	1.329268
Non-weird	168	3	3	7	48	0.285714	0.023810	480	2.857143	21	1.714286

Non-weird	170	1	5	17	64	0.376471	0.005882	324	1.905882	22	0.400000
Non-weird	172	1	2	43	42	0.244186	0.005814	396	2.302326	23	1.325581
Non-weird	174	1	3	29	56	0.321839	0.005747	360	2.068966	24	0.781609
Non-weird	176	3	2	11	40	0.227273	0.022727	540	3.068182	25	2.159091
Non-weird	182	1	7	13	72	0.395604	0.005495	336	1.846154	26	0.263736
Non-weird	184	2	2	23	44	0.239130	0.010870	504	2.739130	27	1.782609
Non-weird	186	1	3	31	60	0.322581	0.005376	384	2.064516	28	0.774194
Non-weird	188	1	2	47	46	0.244681	0.005319	432	2.297872	29	1.319149
Non-weird	190	1	5	19	72	0.378947	0.005263	360	1.894737	30	0.378947
Non-weird	196	2	7	7	72	0.367347	0.010204	448	2.285714	31	0.816327
Non-weird	200	3	5	5	64	0.320000	0.020000	540	2.700000	32	1.420000
Non-weird	204	2	3	17	64	0.313725	0.009804	504	2.470588	33	1.215686
Non-weird	208	3	2	13	48	0.230769	0.019231	630	3.028846	34	2.105769
Non-weird	212	1	2	53	52	0.245283	0.004717	486	2.292453	35	1.311321
Non-weird	220	2	5	11	80	0.363636	0.009091	504	2.290909	36	0.836364
Non-weird	222	1	3	37	72	0.324324	0.004505	456	2.054054	37	0.756757
Non-weird	228	2	3	19	72	0.315789	0.008772	560	2.456140	38	1.192982
Non-weird	230	1	5	23	88	0.382609	0.004348	432	1.878261	39	0.347826
Non-weird	232	2	2	29	56	0.241379	0.008621	630	2.715517	40	1.750000
Non-weird	236	1	2	59	58	0.245763	0.004237	540	2.288136	41	1.305085
Non-weird	238	1	7	17	96	0.403361	0.004202	432	1.815126	42	0.201681
Non-weird	242	1	11	11	100	0.413223	0.004132	432	1.785124	43	0.132231
Non-weird	244	1	2	61	60	0.245902	0.004098	558	2.286885	44	1.303279
Non-weird	246	1	3	41	80	0.325203	0.004065	504	2.048780	45	0.747967
Non-weird	248	2	2	31	60	0.241935	0.008065	672	2.709677	46	1.741935
Non-weird	258	1	3	43	84	0.325581	0.003876	528	2.046512	47	0.744186
Non-weird	260	2	5	13	96	0.369231	0.007692	588	2.261538	48	0.784615
Non-weird	264	3	3	11	80	0.303030	0.015152	720	2.727273	49	1.515152
Non-weird	266	1	7	19	108	0.406015	0.003759	480	1.804511	50	0.180451
Non-weird	268	1	2	67	66	0.246269	0.003731	612	2.283582	51	1.298507
Non-weird	272	3	2	17	64	0.235294	0.014706	810	2.977941	52	2.036765
Non-weird	276	2	3	23	88	0.318841	0.007246	672	2.434783	53	1.159420
Non-weird	280	3	5	7	96	0.342857	0.014286	720	2.571429	54	1.200000
Non-weird	282	1	3	47	92	0.326241	0.003546	576	2.042553	55	0.737589
Non-weird	284	1	2	71	70	0.246479	0.003521	648	2.281690	56	1.295775
Non-weird	286	1	11	13	120	0.419580	0.003497	504	1.762238	57	0.083916
Type of n	**N**	**k**	**p**	**q**	***φ*(n)**	**T(n)**	**| E *φ*|**	***σ* (n)**	**S(n)**	**| E*σ*|**	**Wz(n)**
Non-weird	292	1	2	73	72	0.246575	0.003425	666	2.280822	59	1.294521
Non-weird	296	2	2	37	72	0.243243	0.006757	798	2.695946	60	1.722973
Non-weird	304	3	2	19	72	0.236842	0.013158	900	2.960526	61	2.013158
Non-weird	308	2	7	11	120	0.389610	0.006494	672	2.181818	62	0.623377
Non-weird	310	1	5	31	120	0.387097	0.003226	576	1.858065	63	0.309677
Non-weird	312	3	3	13	96	0.307692	0.012821	840	2.692308	64	1.461538
Non-weird	316	1	2	79	78	0.246835	0.003165	720	2.278481	65	1.291139

Non-weird	318	1	3	53	104	0.327044	0.003145	648	2.037736	66	0.729560
Non-weird	322	1	7	23	132	0.409938	0.003106	576	1.788820	67	0.149068
Non-weird	328	2	2	41	80	0.243902	0.006098	882	2.689024	68	1.713415
Non-weird	332	1	2	83	82	0.246988	0.003012	756	2.277108	69	1.289157
Non-weird	338	1	13	13	144	0.426036	0.002959	588	1.739645	70	0.035503
Non-weird	340	2	5	17	128	0.376471	0.005882	756	2.223529	71	0.717647
Non-weird	344	2	2	43	84	0.244186	0.005814	924	2.686047	72	1.709302
Non-weird	348	2	3	29	112	0.321839	0.005747	840	2.413793	73	1.126437
Non-weird	354	1	3	59	116	0.327684	0.002825	720	2.033898	74	0.723164
Non-weird	356	1	2	89	88	0.247191	0.002809	810	2.275281	75	1.286517
Non-weird	364	2	7	13	144	0.395604	0.005495	784	2.153846	76	0.571429
Non-weird	366	1	3	61	120	0.327869	0.002732	744	2.032787	77	0.721311
Non-weird	368	3	2	23	88	0.239130	0.010870	1080	2.934783	78	1.978261
Non-weird	370	1	5	37	144	0.389189	0.002703	684	1.848649	79	0.291892
Non-weird	372	2	3	31	120	0.322581	0.005376	896	2.408602	80	1.118280
Non-weird	374	1	11	17	160	0.427807	0.002674	648	1.732620	81	0.021390
Non-weird	376	2	2	47	92	0.244681	0.005319	1008	2.680851	82	1.702128
Non-weird	380	2	5	19	144	0.378947	0.005263	840	2.210526	83	0.694737
Non-weird	388	1	2	97	96	0.247423	0.002577	882	2.273196	84	1.283505
Non-weird	392	3	7	7	144	0.367347	0.010204	960	2.448980	85	0.979592
Non-weird	402	1	3	67	132	0.328358	0.002488	816	2.029851	86	0.716418
Non-weird	404	1	2	101	100	0.247525	0.002475	918	2.272277	87	1.282178
Non-weird	406	1	7	29	168	0.413793	0.002463	720	1.773399	88	0.118227
Non-weird	408	3	3	17	128	0.313725	0.009804	1080	2.647059	89	1.392157
Non-weird	410	1	5	41	160	0.390244	0.002439	756	1.843902	90	0.282927
Non-weird	412	1	2	103	102	0.247573	0.002427	936	2.271845	91	1.281553
Non-weird	418	1	11	19	180	0.430622	0.002392	720	1.722488	92	0.000000
Non-weird	424	2	2	53	104	0.245283	0.004717	1134	2.674528	93	1.693396
Non-weird	426	1	3	71	140	0.328638	0.002347	864	2.028169	94	0.713615
Non-weird	428	1	2	107	106	0.247664	0.002336	972	2.271028	95	1.280374
Non-weird	430	1	5	43	168	0.390698	0.002326	792	1.841860	96	0.279070
Non-weird	434	1	7	31	180	0.414747	0.002304	768	1.769585	97	0.110599
Non-weird	436	1	2	109	108	0.247706	0.002294	990	2.270642	98	1.279817
Non-weird	438	1	3	73	144	0.328767	0.002283	888	2.027397	99	0.712329
Non-weird	440	3	5	11	160	0.363636	0.009091	1080	2.454545	100	1.000000
Non-weird	442	1	13	17	192	0.434389	0.002262	756	1.710407	101	-0.027149

Tables 3: Weird $n = 2^k p\, q_1 q_2$

Type of n	n	k	p	q_1	q_2	φ(n)	T (n)	| Eφ |	σ(n)	S (n)	| Eσ |	W z (n)
Weird	836	1	2	11	19	180	0.21531	0.037081	2160	2.58373	0.11842	1.72249
Weird	4030	1	5	13	31	1440	0.35732	0.011911	8064	2.00099	0.03722	0.57171
Weird	5830	1	5	11	53	2080	0.35678	0.011664	11664	2.00069	0.03602	0.57358
Weird	7192	2	2	29	31	1680	0.23359	0.016963	20160	2.80311	0.06132	1.86874
Weird	7912	2	2	23	43	1848	0.23357	0.016936	22176	2.80283	0.06105	1.86855

Weird	9272	2	2	19	6 1	2160	0.23296	0.017472	26040	2.80846	0.06266	1.87662
Weird	10430	1	5	7	149	3552	0.34056	0.015340	21600	2.07095	0.0466	0.70872
Weird	10570	1	5	7	151	3600	0.34059	0.015326	21888	2.07077	0.04655	0.70842
Weird	10792	2	2	19	7 1	2520	0.23351	0.016864	30240	2.80208	0.06032	1.86805

Table 4: Non- Weird $n = 2^k p\ q_1 q_2$

Type of n	n	k	p	q_1	q_2	φ(n	T(n)	| E φ |	σ (n)	S(n)	| Eσ |	Wz(n)
Non-weird	16	1	2	2	2	1	0.0625	0.312500	81	5.0625	1.3125	4.8125
Non-weird	24	1	2	2	3	2	0.08333	0.250000	108	4.5	1	4.16667
Non-weird	40	1	2	2	5	4	0.1	0.200000	162	4.05	0.75	3.65
Non-weird	56	1	2	2	7	6	0.10714	0.178571	216	3.85714	0.64286	3.42857
Non-weird	88	1	2	2	11	10	0.11364	0.159091	324	3.68182	0.54545	3.22727
Non-weird	104	1	2	2	13	12	0.11538	0.153846	378	3.63462	0.51923	3.17308
Non-weird	136	1	2	2	17	16	0.11765	0.147059	486	3.57353	0.48529	3.10294
Non-weird	152	1	2	2	19	18	0.11842	0.144737	540	3.55263	0.47368	3.07895
Non-weird	184	1	2	2	23	22	0.11957	0.141304	648	3.52174	0.45652	3.04348
Non-weird	232	1	2	2	29	28	0.12069	0.137931	810	3.49138	0.43966	3.00862
Non-weird	248	1	2	2	31	30	0.12097	0.137097	864	3.48387	0.43548	3
Non-weird	296	1	2	2	37	36	0.12162	0.135135	1026	3.46622	0.42568	2.97973
Non-weird	328	1	2	2	41	40	0.12195	0.134146	1134	3.45732	0.42073	2.96951
Non-weird	344	1	2	2	43	42	0.12209	0.133721	1188	3.45349	0.4186	2.96512
Non-weird	376	1	2	2	47	46	0.12234	0.132979	1296	3.44681	0.41489	2.95745
Non-weird	424	1	2	2	53	52	0.12264	0.132075	1458	3.43868	0.41038	2.94811
Non-weird	472	1	2	2	59	58	0.12288	0.131356	1620	3.4322	0.40678	2.94068
Non-weird	488	1	2	2	61	60	0.12295	0.131148	1674	3.43033	0.40574	2.93852
Non-weird	536	1	2	2	67	66	0.12313	0.130597	1836	3.42537	0.40299	2.93284
Non-weird	568	1	2	2	71	70	0.12324	0.130282	1944	3.42254	0.40141	2.92958
Non-weird	584	1	2	2	73	72	0.12329	0.130137	1998	3.42123	0.40068	2.92808
Non-weird	632	1	2	2	79	78	0.12342	0.129747	2160	3.41772	0.39873	2.92405
Non-weird	664	1	2	2	83	82	0.12349	0.129518	2268	3.41566	0.39759	2.92169
Non-weird	712	1	2	2	89	88	0.1236	0.129213	2430	3.41292	0.39607	2.91854
Non-weird	776	1	2	2	97	96	0.12371	0.128866	2646	3.40979	0.39433	2.91495
Non-weird	808	1	2	2	101	100	0.12376	0.128713	2754	3.40842	0.39356	2.91337
Non-weird	824	1	2	2	103	102	0.12379	0.128641	2808	3.40777	0.3932	2.91262
Non-weird	856	1	2	2	107	106	0.12383	0.128505	2916	3.40654	0.39252	2.91121
Non-weird	872	1	2	2	109	108	0.12385	0.128440	2970	3.40596	0.3922	2.91055
Non-weird	904	1	2	2	113	112	0.12389	0.128319	3078	3.40487	0.39159	2.90929
Non-weird	1016	1	2	2	127	126	0.12402	0.127953	3456	3.40157	0.38976	2.90551
Non-weird	1048	1	2	2	131	130	0.12405	0.127863	3564	3.40076	0.38931	2.90458
Non-weird	1096	1	2	2	137	136	0.12409	0.127737	3726	3.39964	0.38869	2.90328
Non-weird	1112	1	2	2	139	138	0.1241	0.127698	3780	3.39928	0.38849	2.90288
Non-weird	1192	1	2	2	149	148	0.12416	0.127517	4050	3.39765	0.38758	2.90101
Non-weird	1208	1	2	2	151	150	0.12417	0.127483	4104	3.39735	0.38742	2.90066
Non-weird	1256	1	2	2	157	156	0.1242	0.127389	4266	3.3965	0.38694	2.89968

Non-weird	1304	1	2	2	163	162	0.12423	0.127301	4428	3.39571	0.3865	2.89877
Non-weird	1336	1	2	2	167	166	0.12425	0.127246	4536	3.39521	0.38623	2.8982
Non-weird	1384	1	2	2	173	172	0.12428	0.127168	4698	3.39451	0.38584	2.8974
Type of n	**n**	**k**	**p**	**q_1**	**q_2**	**φ(n**	**T(n)**	**| E φ|**	**σ (n)**	**S(n)**	**| Eσ|**	**Wz(n)**
Non-weird	1448	1	2	2	181	180	0.12431	0.127072	4914	3.39365	0.38536	2.89641
Non-weird	1528	1	2	2	191	190	0.12435	0.126963	5184	3.39267	0.38482	2.89529
Non-weird	1544	1	2	2	193	192	0.12435	0.126943	5238	3.39249	0.38472	2.89508
Non-weird	1576	1	2	2	197	196	0.12437	0.126904	5346	3.39213	0.38452	2.89467
Non-weird	1592	1	2	2	199	198	0.12437	0.126884	5400	3.39196	0.38442	2.89447
Non-weird	1688	1	2	2	211	210	0.12441	0.126777	5724	3.391	0.38389	2.89336
Non-weird	1784	1	2	2	223	222	0.12444	0.126682	6048	3.39013	0.38341	2.89238
Non-weird	1816	1	2	2	227	226	0.12445	0.126652	6156	3.38987	0.38326	2.89207
Non-weird	1832	1	2	2	229	228	0.12445	0.126638	6210	3.38974	0.38319	2.89192
Non-weird	1864	1	2	2	233	232	0.12446	0.126609	6318	3.38948	0.38305	2.89163
Non-weird	1912	1	2	2	239	238	0.12448	0.126569	6480	3.38912	0.38285	2.89121
Non-weird	1928	1	2	2	241	240	0.12448	0.126556	6534	3.389	0.38278	2.89108
Non-weird	2008	1	2	2	251	250	0.1245	0.126494	6804	3.38845	0.38247	2.89044
Non-weird	2056	1	2	2	257	256	0.12451	0.126459	6966	3.38813	0.3823	2.89008
Non-weird	2104	1	2	2	263	262	0.12452	0.126426	7128	3.38783	0.38213	2.88973
Non-weird	2152	1	2	2	269	268	0.12454	0.126394	7290	3.38755	0.38197	2.88941
Non-weird	2168	1	2	2	271	270	0.12454	0.126384	7344	3.38745	0.38192	2.8893
Non-weird	2216	1	2	2	277	276	0.12455	0.126354	7506	3.38718	0.38177	2.88899
Non-weird	2248	1	2	2	281	280	0.12456	0.126335	7614	3.38701	0.38167	2.88879
Non-weird	2264	1	2	2	283	282	0.12456	0.126325	7668	3.38693	0.38163	2.88869
Non-weird	2344	1	2	2	293	292	0.12457	0.126280	7938	3.38652	0.3814	2.88823
Non-weird	2456	1	2	2	307	306	0.12459	0.126221	8316	3.38599	0.38111	2.88762
Non-weird	2488	1	2	2	311	310	0.1246	0.126206	8424	3.38585	0.38103	2.88746
Non-weird	2504	1	2	2	313	312	0.1246	0.126198	8478	3.38578	0.38099	2.88738
Non-weird	2536	1	2	2	317	316	0.12461	0.126183	8586	3.38565	0.38091	2.88722
Non-weird	2648	1	2	2	331	330	0.12462	0.126133	8964	3.3852	0.38066	2.88671
Non-weird	2696	1	2	2	337	336	0.12463	0.126113	9126	3.38501	0.38056	2.8865
Non-weird	2776	1	2	2	347	346	0.12464	0.126081	9396	3.38473	0.3804	2.88617
Non-weird	2792	1	2	2	349	348	0.12464	0.126074	9450	3.38467	0.38037	2.8861
Non-weird	2824	1	2	2	353	352	0.12465	0.126062	9558	3.38456	0.38031	2.88598
Non-weird	2872	1	2	2	359	358	0.12465	0.126045	9720	3.3844	0.38022	2.88579
Non-weird	2936	1	2	2	367	366	0.12466	0.126022	9936	3.3842	0.38011	2.88556
Non-weird	2984	1	2	2	373	372	0.12466	0.126005	10098	3.38405	0.38003	2.88539
Non-weird	3032	1	2	2	379	378	0.12467	0.125989	10260	3.38391	0.37995	2.88522
Non-weird	3064	1	2	2	383	382	0.12467	0.125979	10368	3.38381	0.3799	2.88512
Non-weird	3112	1	2	2	389	388	0.12468	0.125964	10530	3.38368	0.37982	2.88496
Non-weird	3176	1	2	2	397	396	0.12469	0.125945	10746	3.3835	0.37972	2.88476
Non-weird	3208	1	2	2	401	400	0.12469	0.125935	10854	3.38342	0.37968	2.88466
Non-weird	3272	1	2	2	409	408	0.12469	0.125917	11070	3.38325	0.37958	2.88447
Non-weird	3352	1	2	2	419	418	0.1247	0.125895	11340	3.38305	0.37947	2.88425

Non-weird	3368	1	2	2	421	420	0.1247	0.125891	11394	3.38302	0.37945	2.8842
Non-weird	3448	1	2	2	431	430	0.12471	0.125870	11664	3.38283	0.37935	2.88399
Non-weird	3464	1	2	2	433	432	0.12471	0.125866	11718	3.38279	0.37933	2.88395
Type of n	**n**	**k**	**p**	**q_1**	**q_2**	**φ(n**	**T(n)**	**| E φ|**	**σ (n)**	**S(n)**	**| Eσ|**	**Wz(n)**
Non-weird	3544	1	2	2	443	442	0.12472	0.125847	11988	3.38262	0.37923	2.88375
Non-weird	3592	1	2	2	449	448	0.12472	0.125835	12150	3.38252	0.37918	2.88363
Non-weird	3656	1	2	2	457	456	0.12473	0.125821	12366	3.38239	0.3791	2.88348
Non-weird	3688	1	2	2	461	460	0.12473	0.125813	12474	3.38232	0.37907	2.88341
Non-weird	3704	1	2	2	463	462	0.12473	0.125810	12528	3.38229	0.37905	2.88337
Non-weird	3736	1	2	2	467	466	0.12473	0.125803	12636	3.38223	0.37901	2.8833
Non-weird	3832	1	2	2	479	478	0.12474	0.125783	12960	3.38205	0.37891	2.88309
Non-weird	3896	1	2	2	487	486	0.12474	0.125770	13176	3.38193	0.37885	2.88296
Non-weird	3928	1	2	2	491	490	0.12475	0.125764	13284	3.38187	0.37882	2.88289
Non-weird	3992	1	2	2	499	498	0.12475	0.125752	13500	3.38176	0.37876	2.88277
Non-weird	4024	1	2	2	503	502	0.12475	0.125746	13608	3.38171	0.37873	2.8827
Non-weird	4072	1	2	2	509	508	0.12475	0.125737	13770	3.38163	0.37868	2.88261
Non-weird	4168	1	2	2	521	520	0.12476	0.125720	14094	3.38148	0.3786	2.88244
Non-weird	4184	1	2	2	523	522	0.12476	0.125717	14148	3.38145	0.37859	2.88241
Non-weird	4328	1	2	2	541	540	0.12477	0.125693	14634	3.38124	0.37847	2.88216
Non-weird	36	1	2	3	3	4	0.11111	0.194444	144	4	0.75	3.55556
Non-weird	60	1	2	3	5	8	0.13333	0.150000	216	3.6	0.55	3.06667
Non-weird	84	1	2	3	7	12	0.14286	0.130952	288	3.42857	0.46429	2.85714
Non-weird	132	1	2	3	11	20	0.15152	0.113636	432	3.27273	0.38636	2.66667
Non-weird	156	1	2	3	13	24	0.15385	0.108974	504	3.23077	0.36538	2.61538
Non-weird	204	1	2	3	17	32	0.15686	0.102941	648	3.17647	0.33824	2.54902
Non-weird	228	1	2	3	19	36	0.15789	0.100877	720	3.15789	0.32895	2.52632
Non-weird	276	1	2	3	23	44	0.15942	0.097826	864	3.13043	0.31522	2.49275
Non-weird	348	1	2	3	29	56	0.16092	0.094828	1080	3.10345	0.30172	2.45977
Non-weird	372	1	2	3	31	60	0.16129	0.094086	1152	3.09677	0.29839	2.45161
Non-weird	444	1	2	3	37	72	0.16216	0.092342	1368	3.08108	0.29054	2.43243
Non-weird	492	1	2	3	41	80	0.1626	0.091463	1512	3.07317	0.28659	2.42276
Non-weird	516	1	2	3	43	84	0.16279	0.091085	1584	3.06977	0.28488	2.4186
Non-weird	564	1	2	3	47	92	0.16312	0.090426	1728	3.06383	0.28191	2.41135
Non-weird	636	1	2	3	53	104	0.16352	0.089623	1944	3.0566	0.2783	2.40252
Non-weird	708	1	2	3	59	116	0.16384	0.088983	2160	3.05085	0.27542	2.39548
Non-weird	732	1	2	3	61	120	0.16393	0.088798	2232	3.04918	0.27459	2.39344
Non-weird	804	1	2	3	67	132	0.16418	0.088308	2448	3.04478	0.27239	2.38806
Non-weird	852	1	2	3	71	140	0.16432	0.088028	2592	3.04225	0.27113	2.38498
Non-weird	876	1	2	3	73	144	0.16438	0.087900	2664	3.0411	0.27055	2.38356
Non-weird	948	1	2	3	79	156	0.16456	0.087553	2880	3.03797	0.26899	2.37975
Non-weird	996	1	2	3	83	164	0.16466	0.087349	3024	3.03614	0.26807	2.37751
Non-weird	1068	1	2	3	89	176	0.16479	0.087079	3240	3.03371	0.26685	2.37453
Non-weird	1164	1	2	3	97	192	0.16495	0.086770	3528	3.03093	0.26546	2.37113
Non-weird	1212	1	2	3	101	200	0.16502	0.086634	3672	3.0297	0.26485	2.36964

Non-weird	1236	1	2	3	103	204	0.16505	0.086570	3744	3.02913	0.26456	2.36893
Non-weird	1284	1	2	3	107	212	0.16511	0.086449	3888	3.02804	0.26402	2.3676
Non-weird	1308	1	2	3	109	216	0.16514	0.086391	3960	3.02752	0.26376	2.36697

11. Discussion and Interpretation of Results

After deriving the classical closed-form expressions for the Eulers function and the divisor sum function when n is a weird number of the form ($2^k \quad p \quad q_i$), both functions were written explicitly in terms of the numerical structure of the integer. This formulation avoids the need for divisor enumeration or repeated exhaustive factorization. To facilitate numerical comparison between integers of different magnitudes, the two functions were normalized by dividing each of them by n. Accordingly, the normalized quantities $T\ (n)$ and $S\ (n)$ were introduced.

Since weird numbers are abundant but not semi perfect the behavior of the normalized quantities $T\ (n)$and $S\ (n)$ for weird numbers exhibits a structural pattern that differs from that observed for non-weird integers within the same numerical class. This observation motivates the idea of combining these two quantities into a single numerical measure that can act as a preliminary filtering criterion. Such a criterion allows the rapid identification of integers that are likely to be weird, with a computational cost significantly lower than direct methods based on testing subsets of divisors. Based on this idea, a linear combination was constructed which leads to a practical and computationally efficient filtering indicator.

Demonstrates that it can serve as an effective numerical screening indicator. Specifically, weird numbers in the given class are usually clustered within a relatively small and stable number range of the numbers that they produce. This activity shows the suggested amount encapsulates an underlying structural difference between weird and non-weird integers of the identical numerical family. Therefore ($W_z(n)$) may be used in practical prelim classification, which provides a significant reduction in search space prior to the application of the full weirdness verification tests, which are typically computationally expensive subset-sum tests. The mathematical facts thus argue in favor of the utility of the proposed filter in the research and observation of strange numbers as a useful first stage screening procedure. In order to justify the suggested approximation and filtering criterion numerical results are tabulated below.

12. Conclusion

This study presents an analytical framework for investigating weird numbers, based on the relationship between Euler's function φ(n) and the divisor-sum function σ(n), and exploiting their multiplicative properties to identify the structural characteristics that distinguish this class of integers. The main objective was to reduce reliance on computationally expensive subset-sum algorithms when verifying semi perfectness, by developing a preliminary filtering mechanism grounded in analytically tractable mathematical properties. The analysis began with integers of the form $n = 2^k pq$ and was then extended to more general structures. Explicit closed-form expressions for φ(n) and σ(n) were derived and transformed into normalized representations, allowing meaningful comparisons between integers of different magnitudes and structural configurations. The principal components and secondary terms of these representations were also examined, explicit error bounds were established for the approximations employed, and the results showed a high degree of accuracy within the investigated structures. One of the most significant contributions is a direct relationship between the normalized functions S(n) and T(n), which led to a new filtering quantity, $W_z\ (n) = S(n) - 4\ T\ (n)$ This quantity is an analytical indicator reflecting structural differences between weird and non-weird numbers. Although it is neither a necessary nor a sufficient condition for weirdness, it serves as a preliminary criterion that guides the search and eliminates less promising candidates, thereby reducing the search space and computational effort. The numerical results supported its effectiveness in distinguishing between the two types at a preliminary

level within the classes considered.

Theoretically, the study deepens the understanding of the relationship between the prime factorization of integers and the behavior of their associated arithmetic functions. Computationally, it represents a step toward more efficient approaches that combine mathematical analysis with numerical testing rather than exhaustive verification. Directions for future research include generalizing the framework to integers with three or more prime factors, incorporating the filter into algorithms that search for large weird numbers, conducting statistical studies of the distribution of filtering values, and exploring whether the approach extends to other classes of integers with distinctive arithmetic properties.

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Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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