# Jordan Centralizer-derivations on Prime Rings

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**Abstract**
This study formulates and explores three classes of additive operators within associative ring theory: centralizer-derivations, Jordan centralizer-derivations, and Jordan triple centralizer-derivations within the framework of associative rings. Our primary objective is to examine the structural interconnections among these classes of additive mappings. Under the explicit hypothesis that R is a prime ring of characteristic different from two (char(R) ≠ 2), we establish our principal result demonstrating that every Jordan centralizer-derivation acting on R coincides with a centralizer-derivation. Furthermore, for any 2-torsion free ring R, we prove that every Jordan centralizer-derivation satisfies the Jordan triple centralizer-derivation identity, alongside deriving several foundational linearization formulas.
**Keywords:** Prime ring, centralizer derivation, Jordan centralizer derivation.

# المشتقات المركزية من النمط جوردان على الحلقات الاولية

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**الخلاصة**
تسلط هذه الورقة البحثية الضوء على تقديم ودراسة ثلاثة اصناف رئيسية من المؤثرات الاضافية ضمن نطاق نظرية الحلقات التجميعية وهي : المشتقات المركزية ، المشتقات المركزية من النمط جوردان ، المشتقات المركزية الثلاثية من النمط جوردان مع تحليل الروابط التنظيمية والهيكلية التي تجمع بينهم وبناءا على الفرضية الاساسية ان الحلقة R تمثل حلقة اولية تجميعية ذات مميز لايساوي اثنين (Char(R)≠0) ، نتوصل الى النتيجة الجوهرية ان كل مشتقة مركزية من النمط جوردان تتطابق بالضرورة مع مشتقة معتادة فضلا عن برهنة ان الحلقة هي خالية من الالتواء الثنائي وان كل مشتقة مركزية من النمط جوردان تفي بمتطابقة المشتقة المركزية الثلاثية من النمط جوردان الى جانب استنتاج مجموعة من صيغ التبسيط والخطية الاساسية الناظمة لهذه المؤثرات.
**الكلمات المفتاحية:** الحلقة الاولية، المشتقة المركزية، المشتقة المركزية من النمط جوردان، المشتقة المركزية الثلاثية من النمط جوردان

## 1. Introduction

Throughout this manuscript, R represents an associative ring with center Z(R). A ring R is termed prime if the condition aRb = {0} forces either a = 0 or b = 0 for all elements a, b ∈ R, whereas R is called semiprime if aRa = {0} implies a = 0 for every a ∈ R. Moreover, R is said to be 2-torsion free whenever 2a = 0 yields a = 0 for all a ∈ R, a property automatically satisfied when char(R) ≠ 2. For any pair of elements, a, b ∈ R, their Lie commutator is defined by [a, b] = ab - ba [2].

Over recent decades, extensive research has focused on characterizing the commutativity and internal structure of prime and semiprime rings endowed with specific additive operators, notably derivations, left derivations, and centralizers [1, 3]. In a seminal contribution, Zalar [3] investigated centralizers in semiprime rings, establishing that every Jordan centralizer on a 2-torsion free semiprime ring is necessarily a centralizer. Subsequent developments have broadened these results to encompass higher derivations, orthogonal derivations, and reverse centralizers across prime rings and Γ-ring structures [4, 5, 6]. Inspired by the structural synergy between left centralizers and derivations in non-commutative ring theory, we unify both operators to formulate the concept of a centralizer-derivation on a ring R, together with its Jordan and Jordan triple generalizations. The core contributions of this paper are organized as follows: we first derive essential linearization identities for Jordan centralizer-derivations (Lemma 2.3 and Lemma 2.5); we next prove that every Jordan centralizer-derivation on a prime ring R with char(R) ≠ 2 is a centralizer-derivation (Theorem 3.3); and finally, we verify that every Jordan centralizer-derivation on a 2-torsion free ring R acts as a Jordan triple centralizer-derivation (Proposition 3.4).

## 2. Centralizer derivations at rings

**Definition2.1:** Let R be an associative ring. An additive mapping

h : R → R (satisfying h(a + b) = h(a) + h(b) for all a, b ∈ R) is defined as:

(i) a centralizer-derivation of R if h(ab) = 2h(a)b + ah(b) holds for all a, b ∈ R;

(ii) a Jordan centralizer-derivation of R if $h(a^2) = 2h(a)a + ah(a)$ holds for all a ∈ R;

(iii) a Jordan triple centralizer-derivation of R if h(aba) = 2h(a)ba + ah(b)a + abh(a) holds for all a, b ∈ R.

**Example 2.2:** Assume $R = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathcal{Z}\right\}$ be a ring, we define h of R by

$$h\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 10a \\ 0 & 0 \end{pmatrix}, \forall \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathcal{R}.$$

Hence h is centralizerderiv on R.

**Lemma 1:** Let h be J. centralizerderiv. of a prime ring R then for all a,b∈R

i)h(𝑒𝔟+𝔟𝑒) = 2h(𝑒) 𝔟 +𝑒h(𝔟) + 2h(𝔟) 𝑒+𝔟h(𝑒).

ii) $h(eбc+cбe)= 2h(e)бc+eh(б)c+ebh(c)+ 2h(c)бe+ch(б)e+cbh(e)$

iii) If Ris commutative and char.(R)≠2 then

$h(eбc)= 2h(e)бc+ eh(б)c + ebh(c)$.

Proof: $h((e+б)\ (e+б)) = 2h(e+б)(e+б) + (e+б)h(e+б)$

$=2(h(e)+h(б))(e+б) +(e+б)(h(e)+h(б))$

$= 2h(e)e+2h(e)б+2h(б)e+2h(б)б + eh(e)+eh(б)+ бh(e)+бh(б)$ …(1)

Furthermore

$h((e+б)(e+б)) = h(e^2+ eб+бe + б^2)$

$= h(e^2) + h(б^2) + h(eб+бe)$

$= 2h(e)e+ eh(e) + 2h(б)б + бh(б) + h(eб+бe)$ …(2)

Comparing (1) and (2) we get

$h(eб+бe) = 2h(e)б +eh(б) + 2h(б)e+бh(e)$.

ii) Replace $e+c$ for $e$ in Def. 2.1

$h((e+c)б(e+c) = 2h(e+c)б(e+c)+ (e+c)h(б)(e+c)+(e+c)бh(e+c)$

$= 2h(e)бe+2h(e)бc+2h(c)бe+2h(c)бc +eh(б)e+eh(б)c+ch(б)e+ch(б)c+$

$eбh(e) +ebh( c)+cbh(e)+cbh(c)$ …(1)

On other side

$h((e+c)б(e+c))= h(eбe+eбc+cбe+cбc)$

$= h(eбc+cбe)+2h(e)бe+2h(c)бc+eh(б)e+eh(б)c+ch(б)e+ch(б)c+ ebh(e)$

$+ebh( c)+cбh(e)+cбh(c)$ …(2)

Comparing (1) and (2) we obtain

$h(eбc+cбe)= 2h(e)бc+eh(б)c+ebh(c)+ 2h(c)бe+ch(б)a+cbh(e)$.

iii) From ii and since R is commutative

$2h(eбc)= 4h(e)бc+2eh(б)c+2ebh(c)$

By 2-tortion free we get the require result.

**Remark 2.2:** Let h be J. centralizerderiv. of R then φ(*e*,ƀ) on R by

$\dot{\varphi}$(*e*,ƀ) = h(*e*ƀ) – 2h(*e*)ƀ- *e*h(ƀ), for all *e*,ƀ∈R.

**Lemma2:** Assume h is J. centralizerderiv. at R then ∀ *e*,ƀ,*c*,*d*∈R.

i)$\dot{\varphi}$(*e*,ƀ) = -$\dot{\varphi}$(ƀ,*e*)

ii) $\dot{\varphi}$(*e*+*c*,) = $\dot{\varphi}$(*e*,ƀ)+ $\dot{\varphi}$(*c*,ƀ)

iii) $\dot{\varphi}$(*e*,+d) = $\dot{\varphi}$(*e*,ƀ) + $\dot{\varphi}$(*e*,d)

Proof:

i)h(*e*ƀ+ƀ*e*) = 2h(*e*)ƀ +*e*h(ƀ) + 2*h*(ƀ)*e*+ƀh(*e*)

h(*e*ƀ)+ h(ƀ*e*) = 2h(*e*)ƀ +*e*h(ƀ) + 2*h*(ƀ)*e*+ƀh(*e*)

h(*e*ƀ) - 2h(*e*)ƀ -*e*h(ƀ) = - h(ƀ*e*) +2h(ƀ)*e*+ƀh(*e*)

φ(*e*,ƀ) = - $\dot{\varphi}$(ƀ,*e*)

ii) $\dot{\varphi}$(*e*+*c*,) = 2*h*(*e*+*c*)ƀ+(*e*+*c*)h(ƀ)

= 2h(*e*)ƀ+2h(*c*)ƀ + *e*h(ƀ) +*c*h(ƀ)

= 2h(*e*)ƀ + *eh*(ƀ) + 2*h*(*c*)ƀ + *c*h(ƀ)

= $\dot{\varphi}$(*e*,ƀ) + $\dot{\varphi}$(*c*,ƀ)

iv) Using itself technic of (ii) .

**Remark2.3:** If R is the ring consequently h be centralizerderiv. of R iff $\dot{\varphi}$(*e*,ƀ)=0, for

All *e*,∈R.

**3. The Scores**

**Proposition 3:** If h be J. centralizerderiv. of a ring R consequently ∀*e*,ƀ,ɾ∈R

$\dot{\varphi}$(*e*,ƀ)[*e*,ƀ]=0

Proof: Let w= *e*ƀɾƀ*e*+ƀ*e*ɾ*e*ƀ

h(w) = h(*e*(ƀɾƀ)*e*) + ƀ(*e*ɾ*e*)ƀ)

= 2h(*e*)ƀɾƀ*e*+ *e*h(ƀ)ɾƀ*e*+ *e*ƀh(ɾ)ƀ*e*+*e*ƀɾh(ƀ)*e* + *e*ƀɾƀh(*e*)+ 2h(ƀ)*e*ɾ*e*ƀ + ƀh(*e*)ɾ*e*ƀ +

ƀ*e*h(ɾ)*e*ƀ + ƀ*e*ɾh(*e*)ƀ +ƀ*e*ɾ*e*h(ƀ) …(1)

On other side

$\dot{\varphi}$ h(w) = h((*e*b)r(b*e*)+(b*e*)r(*e*b))

= 2h(*e*b)rb*e* +*e*bh(r)b*e* + *e*brh(b*e*) + 2h(b*e*)r*e*b + b*e*h(r)*e*b + b*e*rh(*e*b) …(2)

Comparing (1) and (2) we obtain

0= (2h(*e*b) – 2h(*e*)b-*e*h(b))rb*e* + (2h(b*e*) – 2h(b)*e* – bh(*e*))r*e*b

= $\dot{\varphi}$(*e*,b) rb*e* + $\dot{\varphi}$(b,*e*)r*e*b

= $\dot{\varphi}$(*e*,b)rb*e*– $\dot{\varphi}$(*e*,b)r*e*b

= $\dot{\varphi}$(*e*,b)r[*e*,b]

**Lemma 4:** let h be J. centralizerderiv. of a ring R then ∀ *e*,b,*c*, *d*, r∈R

$\dot{\varphi}$(*e*,b) $r$ [$c$ , $d$] =0

Proof: Substitute *e*+*c* by $a$ $in$ above result we acquire

$\dot{\varphi}$( +*c*,b)r[*e*+*c*,b] = 0

$\dot{\varphi}$(*e*,b)r[*e*,b] + $\dot{\varphi}$(*e*,b)r[*c*,b] + $\dot{\varphi}$(*c*,b)r[*e*,b] + $\dot{\varphi}$(*c*,b)r[*c*,b] =0

$\dot{\varphi}$(*e*,b)[*c*,b] + $\dot{\varphi}$(*c*,)[*e*,b] = 0

hence

0= $\dot{\varphi}$(*e*,)[*c*,b]r $\dot{\varphi}$(*e*,b)r[*c*,b]

= - $\dot{\varphi}$(*e*,b)r[*c*,b]r $\dot{\varphi}$(*c*,b)r[*e*,b]

$\dot{\varphi}$(*e*,b)[*c*, b] = 0 since R is prime (1)

Substitute b+d by b in above result

$\dot{\varphi}$(*e*,+d)[*e*,+d] = 0

$\dot{\varphi}$(*e*,b)r[*e*,b] + $\dot{\varphi}$(*e*,b)r[*e*,d] + $\dot{\varphi}$(*e*,d)r[$a$,b] + $\dot{\varphi}$(*e*,d)r[*e*,d] =0

$\dot{\varphi}$(*e*,b)[*e*,d] + $\dot{\varphi}$(*e*,d)r[*e*,b] =0 [Lemma3]

0 = $\dot{\varphi}$(*e*,)[*e*,d] r $\dot{\varphi}$(*e*,b)r[*e*,d]

= - $\dot{\varphi}$(*e*,)[*e*,d] r $\dot{\varphi}$(*e*,d)r[*e*,b]

$\dot{\varphi}$(*e*,b)r[*e*,d] =0 since R is prime (2)

Now, $\dot{\varphi}(e,)[e+c,b+d] =0$

$\dot{\varphi}\ (e,b)r[e,b] + \dot{\varphi}(e,b)r[e,d] + \dot{\varphi}(e,b)r[c,b] + \dot{\varphi}(e,b)r[c,d] = 0$

Then by Lemma3, (1) and (2) we get

$\dot{\varphi}(e,b)[c, d ] = 0$

**Theorem 5:** Each J. centralizerderiv. of prime ring R with char.R≠2 is centralizerderiv. of R.

Proof: Assume h be J. centralizerderiv. on R by prime of R then Lemma 4 we acquire

$\dot{\varphi}(e,b) =0$ or [ , d]= 0, $\forall\ e,b,c, d\ \in R$.

whether $[c ,d]\ \neq\ 0, \forall c, d \in R\ \Rightarrow\ \dot{\varphi}(e,b) = 0$ , $\forall e,b\in$R, and by 2.3 we acquire h is centralizerderiv.

whether $[c, d] = 0$ , $\forall c,d\in$R, consequently R be comm. ring also Lemma1 we acquire

$h(2eb) = 2\ (2h(e)b + eh(b))$

**Proposition 6:**Each J. centralizerderiv. on ring R with char.R≠2,is J. triple centralizerderiv. of R.

Proof: Replace $eb+ be$ by b in Lemma 1 we acquire

$$h(e(eb+be)+(eb+be)e) = 2h(e)(eb+be) + eh(eb+be) + 2h(eb+be)e+(eb+be)h(e)$$

$$= 2h(e)eb + 2h(e)be + eh(e)b + eeh(b) + eh(b)e + ebh(b) +$$

$$2h(e)be + 2eh(b)e + 2h(b)ee + 2bh(e)e + ebh(e) + beh(e) \qquad (1)$$

On other side

$$h(e(eb+be)+(eb+be)e) = h\ (eeb + ebe + ebe + bee)$$

$$= h\ (eeb) + h\ (bee) + h(\ ebe+ebe)$$

$$= 2h(e)eb + eh(e)b + eeh(b) + 2h(b)ee + bh(e)e + beh(e) + h(2\ ebe) \qquad (2)$$

Comparing (1) and (2) we acquire

$h(2\ ebe) = 2(2h(e)be + eh(b)e + ebh(e))$

Since char.R≠2, we acquire that h is a Jordan triple centralizer deriv. of R.

**Corollary 7:** Each centralizer deriv. on ring R with char. R≠2 is a J-triple centralizer deriv. of R.

Proof: Assume h is a centralizer deriv. on R by Def. 2.1, we obtain that h is a Jordan centralizer derivation of R,

By Prop. 6, we acquire that h is a J-triple centralizer derivation of R.

## 3. Conclusion

In non-commutative algebra it is considered the concept centralizerderiv important concepts. It is a result of merging the concepts of centering and derivation on rings to form the new concept, and its properties were studied. Jordan centralizer, Jordan triple centralizer, and centralizer: the relationships between them have been studied

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